amplitudes as the frequency increases, the amplitude of the first antisymmetric mode is almost an order of magnitude higher than for the first symmetric vibration mode.

Therefore, the influence of the fluid on the vibrations of a bounded plate manifests itself not only in a reduction in the resonance frequencies, but also in the distortion of the resonance plate vibration modes that exerts a substantial influence on the deflection amplitude and the acoustic pressure in the medium.

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Translated by M.D.F.

PMM U.S.S.R.,Vol.50,No.3,pp.335-341,1986
Printed in Great Britain

0021-8928/86 $10.00+0.00

HIGH-FREQUENCY LONGITUDINAL VIBRATIONS OF ELASTIC RODS*

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One-dimensional equations are constructed for the high-frequency longitudinal vibrations of elastic rods. Problems in a section are formulated to determine the "effective" elastic characteristics of the rod. In the case of a circular rod, the elastic characteristics, dispersion curve, and spectrum are found. Comparisons are made with analogous results of the three-dimensional theory of elasticity and with experiment.

1. One dimensional theory of high-frequency longitudinal vibrations of rods. We consider an isotropic homogeneous straight rod of length $2L$ with a constant cross-section S that occupies a volume V in the non-deformed state in the Cartesian coordinate system $x^1, x^2, x^3 \equiv x$ (the superscript 3 is usually omitted). We place the origin of the coordinate system at the centre of the rod and direct the x axis along its central axis. We will assume the cross-section to be centrally symmetric (if $(x^1, x^2) \in S$, then $(-x^1, -x^2) \in S$).

Under given initial conditions the rod performs vibrational motion. The problem is to construct a one-dimensional dynamic model of the rod high-frequency vibrations that is asymptotically exact in the long-wave domain, and is moreover qualitatively descriptive of the rod integral characteristics in the short-wave domain. Taking a variational approach as a basis /1, 2/, we postulate that the rod motion will occur in conformity with the following variational principle

$$\delta \int_{t_0}^{t_1} \int_{-L}^{L} \Lambda\, dx\, dt = 0, \quad \Lambda = K - \Phi \tag{1.1}$$

where K and Φ are the one-dimensional kinetic and internal energy densities of the rod.

The formulas

$$K = {}^1/_2 \rho u_{,t}^2, \quad \Phi = {}^1/_2 E u_{,x}^2 \tag{1.2}$$

turn out to be true in the classical theory of longitudinal vibrations of a rod, where u is the longitudinal displacement averaged over the cross-section, E is Young's modulus, and ρ is the density of the elastic material of the rod. The model (1.1), (1.2) describes the low-frequency, long-wave vibrations of the rod. It is natural to assume that as the vibration frequency increases the internal degrees of freedom that characterise the new modes (branches) of the rod vibrations will become substantial and these vibrations can be described, in a certain frequency range, by eliminating an appropriate set of internal degrees of freedom in the number of arguments of the functions K and Φ. Within the framework of this approach it is most important to determine the set of essential degrees of freedom and to set up the

*Prikl.Matem.Mekhan.,50,3,443-450,1986

dependences of K and Φ on them /1, 2/. The crux of the proposed model of the longitudinal vibrations of a rod is

$$K = {}^1/_2\rho\,(u_{,t}{}^2 + \psi_{,t}{}^2 + v_{,t}{}^2) \tag{1.3}$$
$$\Phi = {}^1/_2\mu\,(s_1 u_{,x}^2 + s_2\psi_{,x}^2 + s_3 v_{,x}^2 + 2r_1 h^{-1}\psi u_{,x} + 2r_2 h^{-1}\psi v_{,x} + 2r_3 h^{-1} v\psi_{,x} + \beta_1{}^2 h^{-2}\psi^2 + \beta_2{}^2 h^{-2} v^2)$$

Here u is the mean longitudinal displacement, ψ and v correspond to two high-frequency branches of the vibrations whose principal terms describe the first natural modes of the cross-section vibrations as a two-dimensional elastic continuum, h is the maximum distance between the boundary S and the centre, and μ is the shear modulus. The kinetic energy density K is simple in form. Unlike quadratic forms of general outline in $u, \psi, v, u_{,x}, \psi_{,x}, v_{,x}$, the function Φ possesses certain features; there are no terms of the type $u^2, \psi_{,x}u, u_{,x}v, uv_{,x}\ldots$ The model (1.1), (1.3) contains eight constants $s_1, s_2, s_3, r_1, r_2, r_3, \beta_1{}^2, \beta_2{}^2$ for whose determination problems in the section, formulated in Sect.2 below, must be solved.

The rod model (1.1), (1.3) is constructed in the same way as the model of high-frequency shell vibrations in /3/ (see /4-6/ also) in two stages. In the first stage, branches of the rod vibrations in the long-wave domain are found from a three-dimensional functional by using a variational-asymptotic method /2/. We hence retain only the principal terms, in the asymptotic sense, in the long-wave domain, and neglect all terms of order h/l as compared with the principal terms (here l is the wavelength along the longitudinal x axis). Representing the rod displacement in the form of the superposition of the vibrations branches found, we take the average of the three-dimensional action function in the second stage and seek the most successful extrapolation to short waves among the asymptotically equivalent functionals. The latter means appending or deleting terms in the average functional that are small in the long-wave domain but are substantial in the short-wave domain. The following criteria underlay the selection of the extrapolation: a) conservation of the principal terms, in the asymptotic sense, of each branch and the principal cross terms between the vibrations branches under consideration in the average functional; b) the hyperbolicity of the averaged equations. These criteria enable us to select the model that will satisfy the mentioned requirements.

2. Long-wave vibration branches. We will first formulate the problem of rod vibrations within the framework of the three-dimensional theory of elasticity. Let $w^i(x^1, x^2, x, t)$ denote displacements of points of the rod. To be specific, we will assume the rod side boundary and endfaces to be load-free. Then the true displacements of the rod are a stationary point of the functional

$$I = \int_{t_0}^{t_1}\int_V \Lambda\, dv\, dt, \quad \Lambda = T - U \tag{2.1}$$
$$T = {}^1/_2\rho w_{i,t} w_{,t}{}^i = {}^1/_2\rho\,(w_{,t}{}^2 + w_{\alpha,t} w_{,t}{}^\alpha)$$
$$U = {}^1/_2\,[\lambda (w_{,\alpha}^\alpha)^2 + 2\lambda w_{,\alpha}^\alpha w_{,x} + (\lambda + 2\mu) w_{,x}^2 + 2\mu w_{(\alpha,\beta)} w^{(\alpha,\beta)} + \mu\,(w_{,\alpha} + w_{\alpha,x})(w^{,\alpha} + w_{,x}^\alpha)]$$

Here, λ, μ are the Lamé coefficients of the elastic material of the rod. The Greek letters correspond to projections on the x^α axis and run through the values 1, 2. The comma in the subscripts denotes partial differentiation, and the parentheses in the subscripts denote the symmetrization operation. Summation is over repeated subscripts.

We perform an asymptotic analysis of the functional (2.1) in the long-wave domain. We make the change of coordinates $\zeta^\alpha = x^\alpha/h$. The small parameter h enters here explicitly in the internal energy density

$$U = {}^1/_2\mu\,[\gamma h^{-2} (w_{|\alpha}{}^\alpha)^2 + 2\gamma h^{-1} w_{|\alpha}{}^\alpha w_{,x} + e^{-2} w_{,x}^2 + 2h^{-2} w_{(\alpha|\beta)} w^{(\alpha|\beta)} + (h^{-1} w_{|\alpha} + w_{\alpha,x})(h^{-1} w^{|\alpha} + w_{,x}^\alpha)], \quad \gamma = \lambda/\mu \tag{2.2}$$
$$e = \sqrt{\mu/(\lambda + 2\mu)}$$

and the action functional (2.1) becomes

$$I = h^2 \int_{t_0}^{t_1} dt \int_{-L}^{L} \langle\Lambda\rangle\, dx \quad \left(\langle\Lambda\rangle = \int_\Omega \Lambda\, d\zeta^1 d\zeta^2\right) \tag{2.3}$$

where Ω is the domain of variation (ζ^1, ζ^2) independent of h, and the vertical bar in the subscripts denotes partial differentiation with respect to ζ^α.

We will assume $h \ll l$, where l is the characteristic scale of variation of the state of stress along the longitudinal coordinate x (the "wavelength") /2/. Keeping the asymptotically principal terms in the functional (2.3), we obtain

$$I = h^2 \int_{t_0}^{t_1} \int_{-L}^{L} \overline{\Lambda}\, dx\, dt \tag{2.4}$$

$$\overline{\Lambda} = {}^1/_2 \langle \rho (w_{,t}{}^2 + w_{\alpha,t} w_{,t}{}^\alpha) - \mu (\gamma h^{-2} (w_{|\alpha}{}^\alpha)^2 + 2h^{-2} w_{(\alpha|\beta)} w^{(\alpha|\beta)} + h^{-2} w_{|\alpha} w^{|\alpha}) \rangle$$

The extrema of the functional (2,4) agree with the extrema of the functional

$$\int_{t_0}^{t_1} \overline{\Lambda}\, dt$$

These are linear superpositions of the eigenmodes (branches) of the vibrations that are split into two series, namely

$$\text{Series} \quad FL_{\parallel}: \ w = v_{(n)} f_{(n)} (\zeta^\alpha), \ w_\alpha \equiv 0 \tag{2.5}$$

$$\text{Series} \quad TFL_{\perp}: \ w_\alpha = \psi_{(n)} f_{\alpha(n)} (\zeta^\beta), \ w \equiv 0$$

The eigenfrequencies and eigenfunctions in each branch can be found by solving the following eigenvalue problem:

$$FL_{\parallel}: \ -\Delta f_{(n)} = \beta^2_{\parallel (n)} f_{(n)}, \ f_{(n)|\alpha} \nu^\alpha = 0 \quad \text{on} \quad \partial\Omega \tag{2.6}$$

$$TFL_{\perp}: \ -(\gamma + 1) f^{\beta}_{(n)|\beta\alpha} - \Delta f_{\alpha(n)} = \beta^2_{\perp (n)} f_{\alpha(n)}$$

$$\gamma f^{\beta}_{(n)|\beta} + 2 f_{(\alpha(n)|\beta)} \nu^\beta = 0 \quad \text{on} \quad \partial\Omega; \ \beta = \omega h / C_2, \ C_2 = \sqrt{\mu/\rho}$$

Here Δ is the Laplace operator, ν^α are the components of the normal vector external to the contour $\partial\Omega$. The following normalization conditions can be imposed on the eigenfunctions $f_{(n)}$ and $f_{\alpha(n)}$:

$$\langle f^2_{(n)} \rangle = 1, \ \langle f_{\alpha(n)} f^{\alpha}_{(n)} \rangle = 1$$

Among the eigenbranches of the vibrations we especially extract the branches comprising the "kernel" of the operators (2.6) (the eigenfrequencies equal zero). They correspond to the classical low-frequency branches of the vibrations and have been studied in detail. We call the remaining branches the high-frequency branches of the vibrations $\omega \to \infty$ as $h \to 0$. We formulate problem (2.6) for these vibrations branches in variational terms

$$FL_{\parallel}: \ \langle f_{|\alpha} \delta f^{|\alpha} \rangle = \beta_{\parallel}{}^2 \langle f \delta f \rangle; \ \langle f^2 \rangle = 1, \ \langle f \rangle = 0 \tag{2.7}$$

$$TFL_{\perp}: \ \langle \gamma f_{|\beta}{}^\beta \delta f_{|\alpha}{}^\alpha + 2 f_{(\alpha|\beta)} \delta f^{(\alpha|\beta)} \rangle = \beta_{\perp}{}^2 \langle f_\alpha \delta f^\alpha \rangle \tag{2.8}$$

$$\langle f_\alpha f^\alpha \rangle = 1, \ \langle f_\alpha \rangle = 0, \ \langle e^{\alpha\beta} f_\alpha \zeta_\beta \rangle = 0$$

where $e^{\alpha\beta}$ are the two-dimensional Levi-Civitȧ symbols. Here and henceforth the number of the branch is omitted on the function.

The following corrections can be found for each branch individually /6/. We will formulate the results in variational terms

$$FL_{\parallel}: \ w = v f (\zeta^\alpha), \ w_\alpha = h v_{,x} g_\alpha (\zeta^\beta) \tag{2.9}$$

$$\langle \gamma g_{|\beta}{}^\beta \delta g_{|\alpha}{}^\alpha + 2 g_{(\alpha|\beta)} \delta g^{(\alpha|\beta)} + \gamma f \delta g_{|\alpha}{}^\alpha - f_{|\alpha} \delta g^\alpha \rangle = \beta_{\parallel}{}^2 \langle g_\alpha \delta g^\alpha \rangle \tag{2.10}$$

$$TFL_{\perp}: \ w_\alpha = \psi f_\alpha (\zeta^\beta), \ w = h \psi_{,x} g (\zeta^\alpha) \tag{2.11}$$

$$\langle g_{|\alpha} \delta g^{|\alpha} + f_\alpha \delta g^{|\alpha} - \gamma f_{|\alpha}{}^\alpha \delta g \rangle = \beta_{\perp}{}^2 \langle g \delta g \rangle \tag{2.12}$$

3. Avarage Lagrangian of an individual high-frequency vibrations branch.

Before turning to the second stage of constructing the equations of the rod vibrations, it is first convenient to find the principal terms of each branch in the long-wave domain in the average action functional. We will examine one branch in the series $FL_{\parallel}$. Let v be an arbitrary function of x and t in (2.9). Substituting (2.9) into the functional (2.3) retaining the principal terms, and taking the average over the section, we obtain

$$\langle \Lambda \rangle = {}^1/_2 \rho (v_{,t}{}^2 + h^2 v^2_{,xt} \langle g_\alpha g^\alpha \rangle) - {}^1/_2 \mu [h^{-2} v^2 \langle f^{|\alpha} f_{|\alpha} \rangle + v^2_{,x} (\langle \gamma (g_{|\alpha}{}^\alpha)^2 + 2 g_{(\alpha|\beta)} g^{(\alpha|\beta)} + 2\gamma f g_{|\alpha}{}^\alpha - 2 g^\alpha f_{|\alpha} \rangle + e^{-2})]$$

According to the variational Eqs.(2.7) and (2.10), we have

$$\langle f_{|\alpha} f^{|\alpha} \rangle = \beta_{\parallel}{}^2$$

$$\langle \gamma (g_{|\alpha}{}^\alpha)^2 + 2 g_{(\alpha|\beta)} g^{(\alpha|\beta)} + \gamma f g_{|\alpha}{}^\alpha - f_{|\alpha} g^\alpha \rangle = \beta_{\parallel}{}^2 \langle g_\alpha g^\alpha \rangle = c \beta_{\parallel}{}^2$$

Therefore

$$\langle\Lambda\rangle = {}^1/_2\rho v_{,t}{}^2 + {}^1/_2\rho ch^2 v^2_{,xt} - {}^1/_2\mu\,[\beta_{\parallel}{}^2 h^{-2}v^2 + v^2_{,x}(c\beta_{\parallel}{}^2 + k_3 + e^{-2})] \tag{3.1}$$
$$k_3 = \langle\gamma f g_{|\alpha}{}^{\alpha} - g^{\alpha} f_{|\alpha}\rangle$$

Since v describes harmonic vibrations in the long-wave domain with a frequency near the frequency $\omega_{\parallel} = \beta_{\parallel}C_2/h$, the term ${}^1/_2\rho chv^2_{,xt}$ in (3.1) can be replaced to a first approximation by ${}^1/_2\mu c\beta_{\parallel}{}^2 v_{,x}{}^2$. Finally, the principal terms of the branch $FL_{\parallel}$ in the average Lagrangian are the following:

$$\langle\Lambda\rangle = {}^1/_2\rho v_{,t}{}^2 - {}^1/_2\mu\,(\beta_{\parallel}{}^2 h^{-2}v^2 + (k_3 + e^{-2})\,v^2_{,x}) \tag{3.2}$$

We have analogously for the branch $TFL_{\perp}$

$$\langle\Lambda\rangle = {}^1/_2\rho\psi_{,t}{}^2 - {}^1/_2\mu\,(\beta_{\perp}{}^2 h^{-2}\psi^2 + (k_2 + 1)\,\psi^2_{,x}) \tag{3.3}$$
$$k_2 = \langle f_{\alpha} g^{|\alpha} - \gamma g f_{|\alpha}{}^{\alpha}\rangle$$

(f_{α} and g are solutions of the variational problems (2.8) and (2.12)).

4. Equations of the rod high-frequency longitudinal vibrations. We shall construct equations describing a family of the first three longitudinal vibration branches. Such a family was selected because of the simplicity of the final formulas. Models including a larger number of interacting vibration branches can be constructed with the same success.

Thus, we represent the rod displacements in the form

$$w = \bar{u}d + \bar{v}f + h\bar{\psi}_{,x}g,\quad w_{\alpha} = \bar{\psi}f_{\alpha} + h\bar{u}_{,x}e_{\alpha} + h\bar{v}_{,x}g_{\alpha} \tag{4.1}$$

where the desired $\bar{u}, \bar{\psi}, \bar{v}$ are arbitrary functions of x, t (the notation without bars is reserved for the functions that appear in the final equations as a result of replacing the desired functions). The function $\bar{u}$ belongs to the classical longitudinal branch of the vibrations and describes the mean longitudinal displacement of the rod (in the long-wave domain). The functions $\bar{\psi}$ and $\bar{v}$ correspond to the first high-frequency longitudinal branches of the vibrations in the series $TFL_{\perp}$ and $FL_{\parallel}$. The basis functions f and f_{α} are orthonormal eigenfunctions in problems (2.7) and (2.8), and the functions g_{α} and g are determined in terms of f and f_{α} in the solution of problems (2.10) and (2.12). It can be shown from an asymptotic analysis of the functional (2.3) that $d = (\mathrm{mes}\,\Omega)^{-1/2} = \mathrm{const}$ while e_{α} is a linear function in ζ^{α}, which is a solution of the variational problem

$$\langle\gamma e_{|\alpha}{}^{\alpha}\delta e_{|\beta}{}^{\beta} + 2e_{(\alpha|\beta)}\delta e^{(\alpha|\beta)} + \gamma d\delta e_{|\alpha}{}^{\alpha}\rangle = 0;\quad \langle e_{\alpha}\rangle = 0,\quad \langle e^{\alpha\beta}e_{\alpha}\zeta_{\beta}\rangle = 0 \tag{4.2}$$

This equation has the obvious solution (ν is Poisson's ratio)

$$e_{\alpha} = -d\nu\zeta_{\alpha} \tag{4.3}$$

We substitute (4.1) into the functional (2.3) and we integrate over the domain Ω. Keeping the principal terms of each branch and the principal cross terms in the average Lagrangian, and taking account of the results in Sect.3, we obtain

$$\begin{aligned}\langle\Lambda\rangle = {}&{}^1/_2\rho(\bar{u}_{,t}{}^2 + 2b_1h\bar{u}_{,t}\bar{\psi}_{,xt} + \bar{\psi}_{,t}{}^2 + 2b_2h\bar{\psi}_{,t}\bar{u}_{,xt} + \\ &2b_3h\bar{\psi}_{,t}\bar{v}_{,xt} + \bar{v}_{,t}{}^2 + 2b_4h\bar{v}_{,t}\bar{\psi}_{,xt}) - {}^1/_2\mu\,[\beta_1{}^2h^{-2}\bar{\psi}^2 + \\ &2a_1h^{-1}\bar{\psi}\bar{u}_{,x} + 2a_2h^{-1}\bar{\psi}\bar{v}_{,x} + 2r_1h^{-1}\bar{u}_{,x}\bar{\psi} + 2r_2h^{-1}\bar{\psi}\bar{v}_{,x} + \\ &2r_3h^{-1}\bar{v}\bar{\psi}_{,x} + \beta_2{}^2h^{-2}\bar{v}^2 + 2a_3h^{-1}\bar{v}\bar{\psi}_{,x} + \\ &k_1\bar{u}^2_{,x} + (k_2 + 1)\,\bar{\psi}^2_{,x} + (k_3 + e^{-2})\,\bar{v}^2_{,x}]\end{aligned} \tag{4.4}$$

Here $\beta_1{}^2$, $\beta_2{}^2$ are the first eigennumbers $\beta_{\perp}{}^2$ and $\beta_{\parallel}{}^2$ in problems (2.8) and (2.7); the formulas for the remaining coefficients have the form

$$\begin{aligned}&b_1 = \langle dg\rangle,\quad b_2 = \langle f_{\alpha}e^{\alpha}\rangle,\quad b_3 = \langle f_{\alpha}g^{\alpha}\rangle,\quad b_4 = \langle fg\rangle \\ &a_1 = \langle\gamma f_{|\alpha}{}^{\alpha}e_{|\beta}{}^{\beta} + 2f_{(\alpha|\beta)}e^{(\alpha|\beta)}\rangle,\quad a_2 = \langle\gamma f_{|\alpha}{}^{\alpha}g_{|\beta}{}^{\beta} + 2f_{(\alpha|\beta)}g^{(\alpha|\beta)}\rangle \\ &a_3 = \langle f_{|\alpha}g^{|\alpha}\rangle,\quad r_1 = \gamma\langle f_{|\alpha}{}^{\alpha}d\rangle,\quad r_2 = \gamma\langle f_{|\alpha}{}^{\alpha}f\rangle \\ &r_3 = \langle f_{|\alpha}f^{\alpha}\rangle,\quad k_1 = E/\mu = 2(1 + \nu)\end{aligned} \tag{4.5}$$

The coefficients k_2, k_3 are given by the formulas in (3.1) and (3.3).

The following equalities can be proved

$$\begin{aligned}&b_1 = b_2,\ b_3 = b_4,\ a_1 = b_2\beta_1{}^2,\ a_2 = b_3\beta_1{}^2,\ a_3 = b_4\beta_2{}^2 \\ &r_1 = -b_2\beta_1{}^2 = -a_1,\ r_3 - r_2 = b_3\,(\beta_1{}^2 - \beta_2{}^2) = a_2 - a_3\end{aligned} \tag{4.6}$$

by using the variational Eqs.(2.7), (2.8), (2.10), (2.12), (4.2).

It follows from relations (4.6) that all the principal cross terms in the average Lagrangian (4.4) form divergent terms in sum, without affecting the equations for $\bar{u}, \bar{\psi}, \bar{v}$. Conseqeuntly, a method of short-wave extrapolation is possible in which all the cross terms in (4.4) are discarded, which would result in independence of the vibration branches. However, additional analysis shows that such extrapolation will result in a qualitatively false description of the dispersion curves and integral characteristics of the rod in the short-wave domain. Consequently, we will here keep all the cross terms and shall only try to seek the replacement of the desired functions, that would simplify (4.4), i.e., would reduce (4.4) to an expression without higher-order derivatives than the first with respect to x and t.

Using properties (4.6), we form complete squares in (4.4)

$$\begin{aligned}\langle\Lambda\rangle = {}^1/_2\rho\,[&(\bar{u}_{,t} + b_2h\bar{\psi}_{,xt})^2 + (\bar{\psi}_{,t} + b_2h\bar{u}_{,xt} + b_3h\bar{v}_{,xt})^2 + \\ &(\bar{v}_{,t} + b_3h\bar{\psi}_{,xt})^2 - (b_2^2 + b_3^2)\,h^2\bar{\psi}^2_{,xt} - b_2^2h^2\bar{u}^2_{,xt} - b_3^2h^2\bar{v}^2_{,xt}] - \\ &{}^1/_2\mu\,[\beta_1^2h^{-2}(\bar{\psi} + b_2h\bar{u}_{,x} + b_3h\bar{v}_{,x})^2 + 2r_1h^{-1}\bar{\psi}\bar{u}_{,x} + \\ &2r_2h^{-1}\bar{\psi}\bar{v}_{,x} + 2r_3h^{-1}\bar{v}\bar{\psi}_{,x} + \beta_2^2h^{-2}(\bar{v} + b_3h\bar{\psi}_{,x})^2 + \\ &(k_1 - b_2^2\beta_1^2)\,\bar{u}^2_{,x} + (k_2 + 1 - b_3^2\beta_2^2)\,\bar{\psi}^2_{,x} + \\ &(k_3 + e^{-2} - b_3^2\beta_1^2)\,\bar{v}^2_{,x}]\end{aligned}$$

The terms in the kinetic energy $-{}^1/_2\rho\,(b_2^2 + b_3^2)\,h^2\bar{\psi}^2_{,xt}$ and $-{}^1/_2\rho b_3^2h^2\bar{v}^2_{,xt}$ at long-waves can be replaced by $-{}^1/_2\mu\,(b_2^2 + b_3^2)\,\beta_1^2\bar{\psi}_{,x}^2$ and $-{}^1/_2\mu b_3^2\beta_2^2\bar{v}^2_{,x}$ (as was done in Sect.3). The term $-{}^1/_2\rho b_2^2h^2\bar{u}^2_{,xt}$ is small at long waves and can be omitted. The Lagrangian then has the form

$$\begin{aligned}\langle\Lambda\rangle = {}^1/_2\rho\,[&(\bar{u} + b_2h\bar{\psi}_{,x})_{,t}{}^2 + (\bar{\psi} + b_2h\bar{u}_{,x} + b_3h\bar{v}_{,x})_{,t}{}^2 + \\ &(\bar{v} + b_3h\bar{\psi}_{,x})_{,t}{}^2] - {}^1/_2\mu\,[\beta_1^2h^{-2}(\bar{\psi} + b_2h\bar{u}_{,x} + b_3h\bar{v}_{,x})^2 + \\ &2r_1h^{-1}\bar{\psi}\bar{u}_{,x} + 2r_2h^{-1}\bar{\psi}\bar{v}_{,x} + 2r_3h^{-1}\bar{v}\bar{\psi}_{,x} + \beta_2^2h^{-2}(\bar{v} + b_3h\bar{\psi}_{,x})^2 + \\ &(k_1 - b_2^2\beta_1^2)\,\bar{u}^2_{,x} + (k_2 - 1 - b_3^2\beta_2^2 + (b_2^2 + b_3^2)\,\beta_1^2)\,\bar{\psi}^2_{,x} + \\ &(k_3 + e^{-2} - b_3^2\beta_1^2 + b_3^2\beta_2^2)\,\bar{v}^2_{,x}]\end{aligned} \tag{4.7}$$

Formula (4.7) shows the substitution $\bar{u} \to u$, $\bar{\psi} \to \psi$, $\bar{v} \to v$, where

$$u = \bar{u} + b_3h\bar{\psi}_{,x},\quad \psi = \bar{\psi} + b_2h\bar{u}_{,x} + b_3h\bar{v}_{,x},\quad v = \bar{v} + b_3h\bar{\psi}_{,x} \tag{4.8}$$

Keeping the principal terms of u, ψ, v and the principal cross terms in (4.7), we obtain that $\langle\Lambda\rangle = K - \Phi$, where K and Φ have the form (1.3) while the coefficients s_1, s_2, s_3 are calculated from the formulas

$$\begin{aligned}&s_1 = k_1 + \frac{r_1^2}{\beta_1^2},\quad s_2 = k_2 + 1 - \frac{r_1^2}{\beta_1^2} + \frac{(r_3 - r_2)^2}{\beta_2^2 - \beta_1^2}\\ &s_3 = k_3 + e^{-2} - \frac{(r_3 - r_2)^2}{\beta_2^2 - \beta_1^2}\end{aligned} \tag{4.9}$$

The variational principle takes the form (1.1) apart from the unimportant constant h^2 By varying the functional (1.1), we obtain the equations of the high-frequency longitudinal rod vibrations and the boundary conditions of the free edge

$$\begin{aligned}&\rho u_{,tt} = \mu\left(s_1u_{,xx} + \frac{r_1}{h}\psi_{,x}\right),\quad \rho\psi_{,tt} = \mu\left(s_2\psi_{,xx} - \frac{r_1}{h}u_{,x} + \right.\\ &\left.\frac{r_3 - r_2}{h}v_{,x} - \frac{\beta_1^2}{h^2}\psi\right),\quad \rho v_{,tt} = \mu\left(s_3v_{,xx} - \frac{r_3 - r_2}{h}\psi_{,x} - \frac{\beta_2^2}{h^2}v\right)\end{aligned} \tag{4.10}$$

$$\begin{aligned}&s_1u_{,x} + \frac{r_1}{h}\psi = 0,\quad s_2\psi_{,x} + \frac{r_3}{h}v = 0,\quad s_3v_{,x} + \frac{r_2}{h}\psi = 0\\ &\text{for}\ \ x = \pm L\end{aligned} \tag{4.11}$$

5. Calculation of the coefficients of the high-frequency longitudinal vibrations equations in the case of a circular rod.

To complete the construction of the model it is necessary to find the coefficients $\beta_1, \beta_2, r_1, r_2, r_3, s_2, s_3$ of Eqs.(4.10) (the coefficient s_1 is found if r_1 and β_1 are known). Finding these coefficients reduces to solving the section problems formulated in Sect.2. As a rule, these can be solved only by numerical methods. One of the exceptions is a rod of circular cross-section where the coefficients are found explicitly. In this case the problem Eqs.(2.7), (2.8), (2.10), (2.12) are solved in a (r, φ) polar coordinate system, where $r = \sqrt{\zeta_\alpha\zeta^\alpha}$, $\varphi = \arg(\zeta^1, \zeta^2)$. For the longitudinal branches of the vibrations obviously $f_\varphi \equiv g_\varphi \equiv 0$ while f, f_r, g, g_r depend only on r (axial symmetry). Therefore, problems (2.7), (2.8), (2.10), (2.12) reduce to boundary value problems for ordinary Bessel differential equations and are solved explicitly in terms of Bessel functions.

Let us present the results. The numbers β_1, β_2 are the least roots of the following transcendental equations

$$e\beta_1J_0(e\beta_1) = 2e^2J_1(e\beta_1),\quad J_1(\beta_2) = 0 \Rightarrow \beta_2 = 3.83171 \tag{5.1}$$

where $J_0(x), J_1(x)$ are Bessel functions of the first kind. The remaining coefficients are given by the formulas

$$\begin{aligned}&r_1 = -\frac{2\gamma\beta_1}{\chi},\quad r_2 = \frac{2\gamma e^2\beta_1^2}{(\beta_2^2 - \beta_1^2e)\,\chi},\quad r_3 = -\frac{2\beta_2^2\beta_1}{(\beta_2^2 - \beta_1^2e^2)\chi}\\ &\chi = \frac{1}{\sqrt{\beta_1^2 - 4(1 - e^2)}}\\ &k_2 + 1 = \frac{\beta_1^2/e^2 - 12}{\chi^2} + \frac{8\beta_1J_0(\beta_1)}{\chi^2J_1(\beta_1)},\qquad k_3 + e^2 = 1 + \frac{8e\beta_2J_1(\beta_2e)}{2e\beta_2J_1(\beta_2e) - \beta_2^2J_0(\beta_2e)}\end{aligned} \tag{5.2}$$

The coefficients s_1, s_2, s_3 are evaluated using (4.9). We present below the values of $\beta_1, r_1, r_2, r_3, s_2, s_3$ as a function of ν

ν	0	0,05	0,1	0,15	0,2	0,25	0,3	0,35
β_1	2,604	2,733	2,885	3,068	3,294	3,584	3,977	4,552
$-r_1$	0	0,262	0,584	0,99	1,52	2,247	3,314	5,057
r_2	0	0,083	0,197	0,355	0,583	0,925	1,474	2,443
$-r_3$	3,097	3,109	3,123	3,138	3,154	3,172	3,192	3,214
s_1	2	2,109	2,241	2,404	2,613	2,893	3,294	3,934
s_2	0,625	0,624	0,622	0,617	0,608	0,591	0,557	0,475
s_3	1,885	1,954	2,032	2,125	2,238	2,382	2,578	2,874

To restore the rod displacements in the values of u, ψ, v it is necessary to know the basis functions f, f_r. We present their expressions

$$f = \frac{J_0(\beta_2 r)}{\sqrt{\pi}\, J_0(\beta_2)}, \quad f_r = \frac{2eJ_1(e\beta_1 r)}{\sqrt{\pi}\chi J_0(e\beta_1)}$$

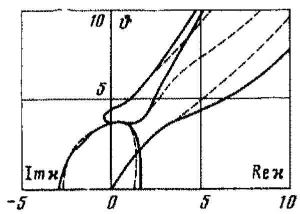

Fig.1

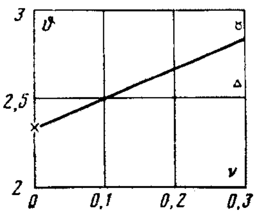

Fig.2

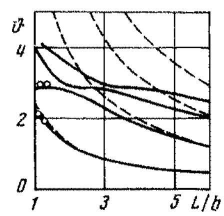

Fig.3

6. Dispersion curves and frequency characteristics of a circular rod.

Dispersion curves are shown in Fig.1 for the set of Eqs.(4.10) examined in the domain $-\infty < x < +\infty$ for $\nu = 0.3317$ (the solid lines). Here $\varkappa = kh$ is the dimensionless wave number, $\vartheta = \omega h / C_2$ is the dimensionless vibration frequency. Its projections on the real and complex planes $(\vartheta, \operatorname{Re}\varkappa)$ and $(\vartheta, \operatorname{Im}\varkappa)$ are drown for the complex branch. We also draw no comparison the first three branches of the dispersion curves calculated by three-dimensional elasticity theory /7/ (the dashed curves). They are solutions of the dispersion equation /8, 9/

$$2p\,(q^2 + \varkappa^2) J_1(p) J_1(q) - (q^2 - \varkappa^2)^2 J_0(p) J_1(q) - 4\varkappa^2 pq J_1(p) J_0(q) = 0 \tag{6.1}$$

where $p = \sqrt{e^2\vartheta^2 - \varkappa^2}$, $q = \sqrt{\vartheta^2 - \varkappa^2}$. By using Newton's method it can be proved that the following asymptotic formulas for the first three vibration branches result from (6.1) for $|\varkappa| \ll 1$ (to $\varkappa^4$ accuracy):

$$\vartheta^2 = 2(1+\nu)\varkappa^2, \quad \vartheta^2 = \beta_1^2 + (k_2+1)\varkappa^2, \quad \vartheta^2 = \beta_2^2 + (k_3 + e^{-2})\varkappa^2 \tag{6.2}$$

where $\beta_1, \beta_2, k_2, k_3$ are given by (5.1) and (5.2). It can be proved that (6.2) also result from the dispersion relationship for (4.10). Therefore, (4.10) describes the behaviour of the dispersion curves in the long-wave domain with asymptotic accuracy. As is seen from Fig.2, qalitatively good agreement with three-dimensional theory is observed in the short-wave domain.

The presence of a complex branch for the dispersion equation of model (4.10) and the interaction of waves with different wave numbers enable us to describe new effects not present in the classical theory of rods. One is the existence of edge modes in a semi-infinite rod. The dependence of the edge mode frequency on ν is shown in Fig.2. For $\nu = 0.29$ we have $\vartheta = 2.83$. For comparison, we present the experimental value of the edge mode frequency $\vartheta_e = 2.92$ (the open circle) (/10/, Table 2). By three-dimensional elasticity theory $\vartheta_t = 2.921$ (the corsses) (/11/, pp.209-210, Fig.86). Experiments and theoretical computations were performed in /10, 11/ for a finite rod: $\nu = 0.29$, $2h = 0.935$ cm, $2L = 8.835$ cm. By the one-dimensional Mindlin-McNiven theory, $\vartheta_M = 2.59$ (the triangle) /12, 13/.

The spectrum for a finite rod is given in Fig.3 for $\nu = 0.29$. The first four eigenfrequencies are shown for the even modes of longitudinal rod vibrations as a function of L/h (the solid lines). The dashed lines denote curves computed by the classical theory of longitudinal rod vibrations. For $\vartheta \geqslant 2$ fairly strong divergence between the classical and constructed theories is observed. Unfortunately, the few experimental data /14/ (the open circles in Fig.3) are insufficient for comparison.

The author is grateful to V.L. Berdichevskii for discussing the research and for valuable remarks.

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Translated by M.D.F.

PMM U.S.S.R.,Vol.50,No.3,pp.341-349,1986
Printed in Great Britain

0021-8928/86 $10.00+0.00

ELASTIC WAVES IN A MATERIAL WITH CHEMOMECHANICAL REACTIONS*

B.N. KLOCHKOV

A theoretical analysis is given of mechanical wave processes in muscle tissue over a broad frequency range. As in /1/, the elastic waves are studied using a continual chemomechanical model /2-5/ extended to the case of an arbitrary discrete and continuous relaxation time spectrum /6/. Analytic expressions containing elastic and viscous parameters, as well as parameters corresponding to the muscle anisotropy and activity, are obtained for the elastic wave velocity and damping in thin muscle tissue specimens. The muscle specimen stability conditions are found. A comparison is made with known experimental results and it is shown that the model constructed describes the elastic-wave characteristics satisfactorily in a muscle in different states.

Investigation of elastic-waves in a medium is an important (often unique) method of determining its structure and rheological and functional properties. This especially concerns media of a biological nature, particularly muscle and internal organ tissues. As a rule, biological media are anisotropic and heterogeneous, where the muscle tissue still manifest active properties, and develops a stress as a result of chemical reactions. During miscle contraction (single, say) the elastic-wave velocity and damping depend on the muscle stress and degree of contraction. Depending on the wavelength, the excitation method, and the propagation

**Prikl.Matem.Mekhan.*,50,3,451-460,1986